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Non-linear interaction of bending deformations of free-oscillating cylindrical shells

V.D. Kubenko*, P.S. Kovalchuk, L.A. Kruk

Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Nesterov st., 3, Kyiv 03057, Ukraine

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Abstract

A method is proposed for the calculation of free oscillations of circular cylindrical shells taking into account the non-linear interaction of their bending deformations. Cases are studied in which a shell is characterized by eigenfrequencies, which are in close proximity or multiple frequencies. Based on analysis of the averaged equations, a number of solutions have been constructed. These solutions are used to investigate the particular qualities of the energy exchange and interaction of the modes of the shell. Phase patterns corresponding to interaction of conjugate forms (2-D model) and forms of various wave-forming parameters (4-D model) are studied. The impact of initial conditions on deformation shapes of free multi-mode-oscillating shells is considered.

1. Introduction

A large number of works are devoted to the analysis of thin-walled cylindrical shells under the geometrically non-linear formulation of their bending vibrations. Systematical reviews of these works are presented in articles by Evensen [1,2], Kubenko and Kovalchuk [3], monographs by Volmir [4], Kubenko et al. [5], Bogdanovich [6], and in other publications.

Usually, simplified one- and two-mode calculation models were used for the investigation of shell constructions. Those models allowed the determination of a number of important, and experimentally confirmed, regularities in the types of non-linear deformation of shells. However, the above-mentioned models are unacceptable for the description of many of the non-linear phenomena based upon the interaction of several forms of bending oscillations. Interconnectivity of these forms creates, under specific 'resonance' conditions, pre-conditions for the realization of

*Corresponding author.
E-mail address: vdk@inmech.kiev.ua (V.D. Kubenko).

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intense energy exchange between various vibration modes, and qualitatively new deformation states of shells. In the scientific literature, cases are described in which a 'latent' internal energy exchange between the bending forms had lead to breakdown situations in some thin-walled objects. Fatigue failure of these objects was preceded by a rather complicated stressed deformed state.

In particular, bending waves of variable parameters were observed travelling in a circumferentially direction, non-stationary cyclic processes of transition from one bending form to another, superposition of several forms (having different wave-forming parameters), etc.

To substantiate these and similar phenomena, it is necessary to develop a non-linear dynamic theory of shells as systems of multiple degrees of freedom that could enable a description of the characteristic features of the energy exchange and interaction between the various forms. Problems of this type, though they have a scientific and applicative significance, are not sufficiently covered in the literature. The initial investigations in this field were first carried out over 30 years ago by Evensen, Dowell, Olson, Fyng, and Ventres [1,7–11] and others, who considered the induced non-linear oscillations, as well as auto-oscillations (flutter), of circular cylindrical shells taking into account the interaction of conjugated (and non-conjugated in the case of flutter) bending forms. Further investigations have been carried out by Matsuzaki and Kobayashi [12], Ginsberg [13], Chen and Babcock [14], and Kubenko et al. [15,16]. In recently published studies by Amabili et al. [17–19] and others, various problems of non-linear dynamics and stability of circular cylindrical shells filled with a liquid (including a mobile one), which can be simulated by systems of multiple degrees of freedom, were discussed. When approximating the dynamical deflections of the shells, conjugated and non-conjugated bending forms were taken into account. This enabled the authors to study the effect of non-linear interaction between the various forms on induced oscillations of the shells, as well as on their dynamical instability conditioned by a mobility factor of the liquid.

The purpose of this work is the investigation of the main regularities of non-linear interaction of both the conjugated and the arbitrary bending forms of free-oscillating circular-cylindrical shells. Cases are considered in which the eigenfrequencies meet definite (resonance) relationships. A method for the calculation of parameters of the shells' multi-mode oscillations (two and four fundamental modes were taken into account) in resonant conditions is proposed. The first solutions of the shells' motions describing the energy-exchange processes among various modes are constructed and analyzed. Phase patterns following from averaged equations of motion of the studied shells are considered.

2. Initial dynamical equations

To describe the process of multi-mode dynamical deformation of a shell, middle-bending equations of the Donnell–Mushtari–Vlasov type [4] of the following form are used:

$$\frac{D}{h}\nabla^4 w = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{1}{R}\frac{\partial^2 \phi}{\partial x^2} - \rho\frac{\partial^2 w}{\partial t^2}, \tag{1a}$$

$$\frac{1}{E}\nabla^4 \phi = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\frac{\partial^2 w}{\partial x^2}. \tag{1b}$$

The definition of the symbols in these equations is given in Appendix D.

Suppose the free resting-on conditions (SS1/SS1) are realized at the end sections of the shell

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad v = 0, \quad N_x = 0 \quad \text{for } x = 0, x = l. \tag{2}$$

To clarify a substantial part of the question of the non-linear interaction of the oscillating shell forms, its deflection is approximated by the four-mode expansion

$$\begin{aligned} w = {} & f_1(t) \cos s_1 y \sin \lambda x + f_2(t) \sin s_1 y \sin \lambda x + f_3(t) \cos s_2 y \sin \lambda x \\ & + f_4(t) \sin s_2 y \sin \lambda x + f_5(t) W_0(x). \end{aligned} \tag{3}$$

Here $f_k(t)$ $(k = 1-4)$ are the unknown time-dependent functions and represent the generalized co-ordinates of the shell; $\lambda = \lambda_m = m\pi/l$ and $s_i = n_i/R$ $(i = 1-2)$ are the wave-formation parameters.

As can be seen, this expansion simultaneously includes forms having both the same and the different wave-forming parameters. The last term of Eq. (3) is introduced to reflect specifically the large-deflection deformation of the shell. Consider the well-known, and confirmed by numerous experiments, effect of "predominantly inwards buckling" [4]. Taking into account Eq. (2), an axially symmetric function $W_0(x)$ is set in the form

$$W_0(x) = \sin^4 \lambda x. \tag{4}$$

Some authors (for example, Volmir [4] and others) represent the function $W_0(x)$ in the form $W_0(x) = \sin^2 \lambda x$. Other authors (for example, Evensen [1,2] and others) determine this function from the periodicity condition of the circumferential displacement v (in this case the function $f_5(t)$ can be expressed in terms of the functions $f_k(t)$, $k = 1-4$). However, the boundary conditions (2) are partly satisfied in both these approaches, because the condition $\partial^2 w/\partial^2 x = 0$ is not satisfied at the end sections of the shell. The function $W_0(x)$ in form (4) satisfies both conditions (2) and the physical effect referred above.

Substituting Eq. (3), and accounting for Eq. (4), in Eq. (1b), the functions of stresses ϕ are defined, supposing that

$$\phi = \phi_p + \phi_h. \tag{5}$$

Here, ϕ_p is a particular solution of this equation, with

$$\begin{aligned} \phi_p = {} & \phi_1 \cos s_1 y \sin \lambda x + \phi_2 \sin s_1 y \sin \lambda x + \phi_3 \cos s_2 y \sin \lambda x \\ & + \phi_4 \sin s_2 y \sin \lambda x + \phi_5 \cos 2\lambda x + \phi_6 \cos 2 s_1 y + \phi_7 \cos 2 s_2 y \\ & + \phi_8 \sin 2 s_1 y + \phi_9 \sin 2 s_2 y + \phi_{10} \cos s_1 y \sin 3\lambda x + \phi_{11} \cos s_2 y \sin 3\lambda x \\ & + \phi_{12} \sin s_1 y \sin 3\lambda x + \phi_{13} \sin s_2 y \sin 3\lambda x + \phi_{14} \cos(s_1 - s_2) y \cos 2\lambda x \\ & + \phi_{15} \cos(s_1 - s_2) y + \phi_{16} \cos(s_1 + s_2) y \cos 2\lambda x + \phi_{17} \cos(s_1 + s_2) y \\ & + \phi_{18} \sin(s_1 - s_2) y \cos 2\lambda x + \phi_{19} \sin(s_1 - s_2) y + \phi_{20} \sin(s_1 + s_2) y \cos 2\lambda x \\ & + \phi_{21} \sin(s_1 + s_2) y + \phi_{22} \cos 4\lambda x + \phi_{23} \cos s_1 y \sin 5\lambda x \\ & + \phi_{24} \sin s_1 y \sin 5\lambda x + \phi_{25} \cos s_2 y \sin 5\lambda x + \phi_{26} \sin s_2 y \sin 5\lambda x. \end{aligned} \tag{6}$$

Here ϕ_j (j = 1–26) are functions of time whose values are listed in Appendix A. Function ϕ_h of Eq. (5) represents by itself a homogeneous solution that has, taking account of conditions (2) and the condition of periodicity of the shell [4], the following form:

$$\phi_h = \frac{E}{16}\left\{[s_1^2(f_1^2 + f_2^2) + s_2^2(f_3^2 + f_4^2)] - \frac{3f_5}{R}\right\}x^2. \tag{7}$$

Then, using the Galerkin method, equations for determination of the unknown functions $f_i(t)$ are derived, which are a part of Eq. (3):

$$\begin{aligned}
&\ddot{f}_1 + \omega_1^2 f_1 + k_{11}(f_1^2 + f_2^2)f_1 + k_{12}(f_3^2 + f_4^2)f_1 + k_{13} f_1 f_5 + k_{14} f_1 f_5^2 = 0,\\
&\ddot{f}_2 + \omega_1^2 f_2 + k_{11}(f_1^2 + f_2^2)f_2 + k_{12}(f_3^2 + f_4^2)f_2 + k_{13} f_2 f_5 + k_{14} f_2 f_5^2 = 0,\\
&\ddot{f}_3 + \omega_2^2 f_3 + k_{21}(f_1^2 + f_2^2)f_3 + k_{22}(f_3^2 + f_4^2)f_3 + k_{23} f_3 f_5 + k_{24} f_3 f_5^2 = 0,\\
&\ddot{f}_4 + \omega_2^2 f_4 + k_{21}(f_1^2 + f_2^2)f_4 + k_{22}(f_3^2 + f_4^2)f_4 + k_{23} f_4 f_5 + k_{24} f_4 f_5^2 = 0,\\
&\ddot{f}_5 + \omega_3^2 f_5 + k_{31}(f_1^2 + f_2^2) + k_{32}(f_3^2 + f_4^2) + k_{33}(f_1^2 + f_2^2)f_5 + k_{34}(f_3^2 + f_4^2)f_5 = 0.
\end{aligned} \tag{8}$$

Values of the frequency parameters ω_i (i = 1, 2, 3) and coefficients k_{jk} at the non-linear terms are given in Appendix B.

Eqs. (8) serve as a base for calculation of the processes of energy exchange and interaction between bending forms corresponding to the wave parameters s_n and λ. The calculation technique essentially depends on the presence or absence of the internal resonances [15,20] in system (8), where system (8) is usually first simplified taking account of the conditions of $f_5(t) \ll f_j(t)(j = 1-4)$ and $\omega_3 \gg \omega_i$ $(i = 1-2)$ [4]. This provides a possibility of determination of the function f_5 from the solution of the quasi-static problem. Supposing, in particular, that $\partial^2(f_5)/\partial t^2$=0, from the last equation of Eq. (8) it is found that

$$f_5 = -\frac{[k_{31}(f_1^2 + f_2^2) + k_{32}(f_3^2 + f_4^2)]}{[\omega_3^2 + k_{33}(f_1^2 + f_2^2) + k_{34}(f_3^2 + f_4^2)]}. \tag{9}$$

Expanding Eq. (9) in a series gives the approximate value of function f_5 as

$$\begin{aligned}
f_5 = &-\frac{1}{\omega_3^2}[k_{31}(f_1^2 + f_2^2) + k_{32}(f_3^2 + f_4^2)] + \frac{1}{\omega_3^4}[k_{31}k_{33}(f_1^2 + f_2^2)^2\\
&+ k_{32}k_{34}(f_3^2 + f_4^2)^2 + (k_{31}k_{34} + k_{32}k_{33})(f_1^2 + f_2^2)(f_3^2 + f_4^2)] + \cdots,
\end{aligned} \tag{10}$$

which characterizes the axial-symmetric item of the w net deflection (3).

Taking into account the above, the initial system of Eqs. (8) can be represented in the form of four equations constructed with respect to the "dominant" generalized co-ordinates f_k (k = 1–4).

3. Interaction of the conjugated forms

Initially, suppose that the shell's eigenfrequencies ω_1 and ω_2 are not in close proximity and not multiple frequencies. In this case, intensive interaction between the forms of the wave parameters

s_1, λ and s_2, λ is absent [15]. At the same time, intense energy exchange will be realized between the conjugated forms of $\cos(s_1y)\sin(\lambda x)$ and $\sin(s_1y)\sin(\lambda x)$, as well as $\cos(s_2y)\sin(\lambda x)$ and $\sin(s_2y) \times \sin(\lambda x)$. Consider characteristic features of such an energy exchange for the forms of wave parameters s_1, λ (the energy exchange for the second pair can be treated analogously).

Expansion (10) is substituted into the first two equations of Eq. (8) and the terms up to the fifth power inclusive with respect to functions f_1 and f_2 are taken in account. It may be noted that the non-linear terms of these equations are small in comparison with the rest of the (linear) terms because all of them are proportional to the small parameter $\varepsilon_0 = w_{max}/R$ [14], where w_{max} is a maximal radial deflection of a shell. Taking this into account, the Bogolyubov–Mitropolsky method [20] may be used to construct the approximate periodic solution for the present equations. In accordance with this method it can be assumed that

$$f_1 = a\cos(\omega t + \vartheta_1),$$
$$f_2 = b\cos(\omega t + \vartheta_2), \quad \omega = \omega_1. \tag{11}$$

In order to determine the parameters of amplitude (a and b) and phase (ϑ_1 and ϑ_2), the following simultaneous equations are obtained:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{ab^2}{8\omega}[\gamma_1 + c_1(a^2 + b^2)]\sin 2\theta,$$

$$\frac{\mathrm{d}b}{\mathrm{d}t} = -\frac{a^2b}{8\omega}[\gamma_1 + c_1(a^2 + b^2)]\sin 2\theta,$$

$$a\frac{\mathrm{d}\vartheta_1}{\mathrm{d}t} = \frac{\gamma_1 a}{8\omega}(3a^2 + 2b^2 + b^2\cos 2\theta) + \frac{c_1 a}{8\omega_1}\left[(a^2 + b^2)(2a^2 + b^2) + \frac{a^4 + b^4}{2} + b^2(2a^2 + b^2)\cos 2\theta\right],$$

$$b\frac{\mathrm{d}\vartheta_2}{\mathrm{d}t} = \frac{\gamma_1 b}{8\omega}(3b^2 + 2a^2 + a^2\cos 2\theta) + \frac{c_1 b}{8\omega_1}\left[(a^2 + b^2)(a^2 + 2b^2) + \frac{a^4 + b^4}{2} + a^2(2b^2 + a^2)\cos 2\theta\right]. \tag{12}$$

Here, the following notations are used:

$$\gamma_1 = k_{11} - \frac{k_{13}k_{31}}{\omega_3^2},$$
$$c_1 = \frac{k_{31}}{\omega_3^4}(k_{13}k_{33} + k_{14}k_{31}),$$
$$\theta = \vartheta_1 - \vartheta_2. \tag{13}$$

From investigation of these equations, the first two of them give a solution of the type

$$a^2 + b^2 = \bar{C}_0^2, \tag{14}$$

where $\bar{C}_0$ is the integration constant. After introduction of a new variable $\xi = a^2/\bar{C}_0^2$, system (12) can be converted, taking account of Eq. (14), to a pair of equations

$$\begin{aligned}\frac{\mathrm{d}\xi}{\mathrm{d}t} &= k\xi(1-\xi)\sin 2\theta,\\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{k}{2}(2\xi - 1)(1-\cos 2\theta).\end{aligned} \tag{15}$$

Here $k = (\gamma_1 + c_1\bar{C}_0^2)\bar{C}_0^2/8\omega$.

Based on these equations, one more solution can be derived:

$$\xi(1-\xi)\sin^2\theta = \bar{C}_1 = const. \tag{16}$$

The solutions obtained describe, in essence, the energy-exchange processes occurring between the conjugated modes. Integral (14) is, in particular, evidence of the competition that takes place between amplitudes a and b of bending forms $\cos(s_1y)\sin(\lambda x)$ and $\sin(s_1y)\sin(\lambda x)$, respectively. Integral (16), in its turn, characterizes a "redistribution" of energy between the amplitude ξ and phase ϑ parameters of the shell.

From Eqs. (15) stationary solutions of type $\xi = \xi_0 = const, \theta = \theta_0 = const$ can be obtained. These solutions correspond to the two following classes:

$$\theta = k\pi, \quad \xi = \xi_0 \quad (0 \leqslant \xi_0 \leqslant 1), \tag{17}$$

$$\theta = \frac{(2k-1)\pi}{2}, \quad \xi = 1/2. \tag{18}$$

Here, $k = 0, \pm 1, \pm 2, \ldots$.

It should be noted that the solutions $\xi = 0$ and 1 correspond to decoupled oscillations of the shell occurring according to the forms of $\cos(s_1y)\sin(\lambda x)$ and $\sin(s_1y)\sin(\lambda x)$. All the other solutions are a superposition of both the forms. To investigate the stability of the equations obtained, qualitative analysis of the singular points of the equation

$$\frac{\mathrm{d}\xi}{\mathrm{d}\theta} = \frac{2\xi(1-\xi)\sin 2\theta}{(2\xi-1)(1-\cos 2\theta)},$$

obtained from the system of Eqs. (15) was performed. Specifically, it was found that the singular points of a "saddle" type correspond to the stationary solutions (17) in the configuration space (ξ, θ). Therefore, these are unstable solutions. The singular points of the "centre" type correspond to solutions (18). Thus, the stable solutions are represented by formula (18); these correspond to deformation of the shell in the form of a circular running wave, as follows from Eq. (3). In fact, in this case, if one does not account for the form of wave parameter s_2, the deflection w is of "wavy" form:

$$w = a_0\cos(s_1y \pm (\omega_1 t + \vartheta_{10}))\sin\lambda x + f_5\sin^4\lambda x \quad (a_0, \vartheta_{10} = const). \tag{19}$$

Note that, to realize the "running" waves (19) it is necessary to give the specified initial conditions, namely the initial amplitudes of the forms $\cos(s_1y)\sin(\lambda x)$ and $\sin(s_1y)\sin(\lambda x)$ must be equal, i.e., $a(0) = b(0)$. Moreover, the phase displacement θ of the exciters of both the forms must be chosen as follows: $\theta = \pm\pi/2$. The stationary solutions are absent for other initial conditions and generally the deflection w will be represented as a superposition of two waves:

$$w = [a(t)\cos(\omega t + \vartheta_1(t))\cos s_1 y + b(t)\cos(\omega t + \vartheta_2(t))\sin s_1 y]\sin \lambda x + f_5(t)\sin^4 \lambda x,$$

where a, b, ϑ_1 and ϑ_2 are the time-dependent functions, which can be obtained from Eq. (12), and the function $f_5(t)$ is found from relationship (10).

It is important to emphasize that the single-mode types of deformation (the "standing-wave" deformation) are impossible in principle in the shell considered as a result of an instability of the solutions $\xi = 0$, $\theta = k\pi$ and $\xi = 1$, $\theta = k\pi$.

Consider a numerical example. Let a shell be characterized by the following parameters:

$$E = 2 \times 10^{11}\ \text{Pa}, \quad \rho = 7.8 \times 10^3\ \text{kg/m}^3, \quad \frac{h}{R} = 3.125 \times 10^{-3},$$

$$l/R = 2.45, \quad R = 0.16\,\text{m}, \quad \mu = 0.3. \tag{20}$$

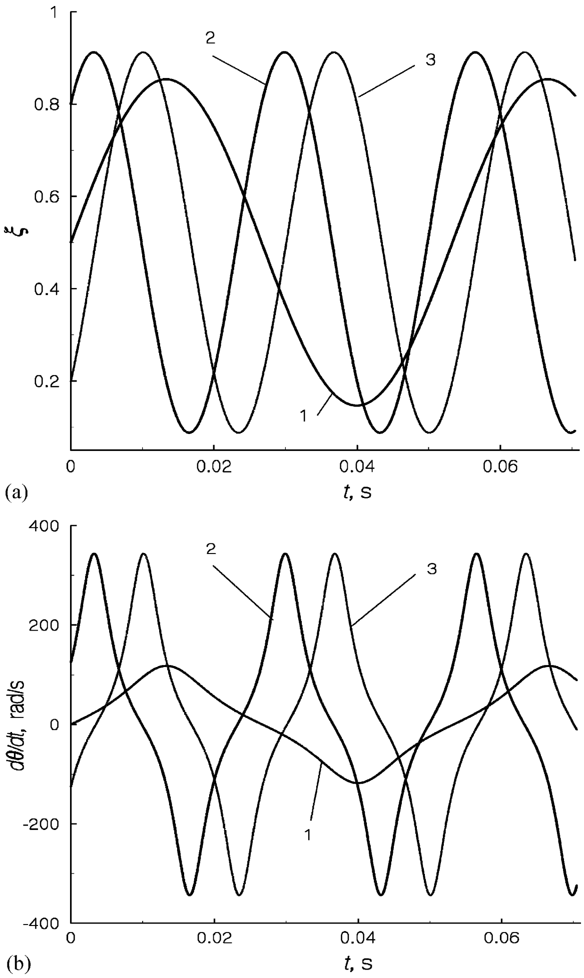


Fig. 1. Amplitude ξ (a) and phase velocity $\mathrm{d}\theta/\mathrm{d}t$ (b) for free vibrations of the shells at: 1—$a(0) = b(0) = h/2$; 2—$a(0) = h$, $b(0) = h/2$; 3—$a(0) = h/2$, $b(0) = h$, when $\theta(0) = \pi/4$.

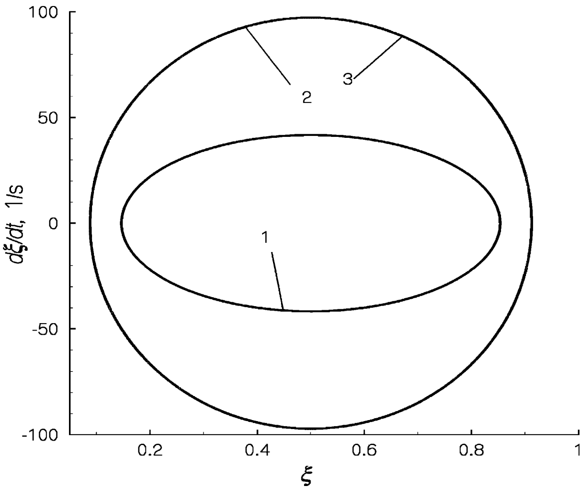


Fig. 2. Phase trajectories of free shell vibrations of $\xi(0) = 0.5$ (curve 1), $\xi(0) = 0.8$ (curve 2), $\xi(0) = 0.2$ (curve 3) at $\theta(0) = \pi/4$.

Minimal eigenfrequency of the shell $\omega_{1\,min} = 283.8$ Hz corresponds to the mode of $m = 1$, $n_1 = 6$. The results of numerical integration of system (15) by the Runge–Kutta method, accounting for Eq. (20) and initial conditions $a(0) = b(0) = h/2$ (curves 1), $a(0) = h$, $b(0) = h/2$ (curves 2), $a(0) = h/2$, $b(0) = h$ (curves 3), and $\theta(0) = \pi/4$ are presented in Fig. 1. As seen from the graphs, the amplitude parameter ξ (Fig. 1a) and phase velocity $\mathrm{d}\theta/\mathrm{d}t$ (Fig. 1b) represent by themselves the oscillating in time functions with their periods being essentially dependent on the levels of imposed initial amplitudes $\xi(0)$. Note that these periods characterize the time of the energy transmitting (full or partial) from one form to another. The energy transmitting process is a "self-controlled" one; in a real shell it will continue up to the moment of exhaustion of the energy, initially added to the shell at the moment of initiation, and spent to compensate for the work of internal and external friction forces. This process will be continuous because in the shell equations (1a,b) no account is taken of damping. Curves 2 and 3 possess the same period of "oscillations" in view of coincidence of their values at $a^2(0) + b^2(0) = \bar{C}_0^2$.

Fig. 2 illustrates the phase patterns of free oscillations of the shell, where curves 1, 2, and 3 are drawn with the same initial data as the corresponding curves in Fig. 1.

4. Interaction of the non-conjugated forms

When considering the general case of the interaction of the forms it would be expedient to represent system (8), using the following substitutions:

$$f_1 = A(t)\cos\alpha(t), \quad f_2 = A(t)\sin\alpha(t),$$
$$f_3 = B(t)\cos\beta(t), \quad f_4 = B(t)\sin\beta(t), \tag{21}$$

in the following form (convenient for the investigation) [5]:

$$\ddot{A}+(\omega_1^2-\dot{\alpha}^2)A+\gamma_1A^3+\delta_1AB^2+c_1A^5+d_1A^3B^2+e_1AB^4=0,\quad A\ddot{\alpha}+2\dot{A}\dot{\alpha}=0,$$
$$\ddot{B}+(\omega_2^2-\dot{\beta}^2)B+\gamma_2B^3+\delta_2BA^2+c_2B^5+d_2B^3A^2+e_2BA^4=0,\quad B\ddot{\beta}+2\dot{B}\dot{\beta}=0. \tag{22}$$

Values of parameters γ_i, δ_i, c_i, d_i, e_i, $(i=1,2)$ are presented in Appendix C. Eqs. (22) were derived taking into account the terms up to the fifth order inclusive with respect to functions A and B, which are defined by relationship (10).

If one takes into account variables (21), deflection w (3) takes a form

$$w=A\cos(s_1y-\alpha)\sin\lambda x+B\cos(s_2y-\beta)\sin\lambda x+C\sin^4\lambda x,\quad (C=f_5), \tag{23}$$

which gives an obvious physical picture about the character of the multi-mode deformation of a free-oscillating shell. This deformation can represent by itself, in particular, a superposition of standing waves (with $\alpha(t)\equiv\alpha_0\equiv const$ and $\beta(t)\equiv\beta_0\equiv const$) or running waves (with $\dot{\alpha}(t)\neq 0, \dot{\beta}(t)\neq 0$). In order to investigate an interaction of these waves, construct approximate periodical solutions of system (22). In so doing, a method set out in Refs. [5,16] will be used.

By analyzing Eqs. (22) by the method in Ref. [20], to a first approximation, the following relations between the resonances were found:

$$\omega_1\approx\omega_2,\quad \omega_1\approx 2\omega_2,\quad \omega_1\approx\omega_2/2. \tag{24}$$

These situations create preconditions for the realization of an intense energy exchange between forms of the shell. Consider the solutions corresponding to these relations.

4.1. The internal resonance $\omega_1\approx\omega_2$

The study of internal resonance is restricted to non-linear terms of the third order in Eqs. (22). Periodical solution of the said equations are represented, in accordance with Ref. [5], in the form

$$A=\sqrt{u_1+v_1\sin\psi_1},\quad \psi_1=2(\omega t+\vartheta_1),$$
$$B=\sqrt{u_2+v_2\sin\psi_2},\quad \psi_2=2(\omega t+\vartheta_2),$$
$$\alpha=\varphi_1+\arctan\frac{(u_1\tan(\psi_1/2)+v_1)}{M_1},$$
$$\beta=\varphi_2+\arctan\frac{(u_2\tan(\psi_2/2)+v_2)}{M_2}. \tag{25}$$

Here, u_i, v_i, φ_i, ϑ_i $(i=1,2)$ are the unknown functions of time that are to be determined from the averaged equations [16]:

$$\frac{du_1}{dt}=-\delta_1\frac{v_1v_2}{2\omega}\sin 2\theta,\quad \frac{du_2}{dt}=\delta_2\frac{v_1v_2}{2\omega}\sin 2\theta,$$

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = -\delta_1 \frac{u_1 v_2}{2\omega} \sin 2\theta, \quad \frac{\mathrm{d}v_2}{\mathrm{d}t} = \delta_2 \frac{u_2 v_1}{2\omega} \sin 2\theta,$$
$$\frac{\mathrm{d}\vartheta_1}{\mathrm{d}t} = \frac{1}{4\omega}\left(3\gamma_1 u_1 + 2\delta_1 u_2 + \delta_1 \frac{v_2 u_1}{v_1} \cos 2\theta\right),$$
$$\frac{\mathrm{d}\vartheta_2}{\mathrm{d}t} = \frac{1}{4\omega}\left(3\gamma_2 u_2 + 2\delta_2 u_1 + 2\Delta_1 + \delta_2 \frac{u_2 v_1}{v_2} \cos 2\theta\right),$$
$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = -\frac{M_1}{4\omega}\left[\gamma_1 + \delta_1 v_2\left(\frac{\cos 2\theta}{v_1} - \frac{\sin 2\theta}{u_1}\right)\right],$$
$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = -\frac{M_2}{4\omega}\left[\gamma_2 + \delta_2 v_1\left(\frac{\cos 2\theta}{v_2} - \frac{\sin 2\theta}{u_2}\right)\right], \tag{26}$$

with $M_i = \sqrt{u_i^2 - v_i^2}$ $(i = 1, 2)$, $\theta = \vartheta_2 - \vartheta_1$, $\Delta_1 = \omega_2^2 - \omega_1^2$, $\omega = \omega_1$.

From the first two equations of this system one can obtain an integral of the form

$$u_1 + u_2 = C_0, \tag{27}$$

where

$$C_0 = \frac{1}{2\omega_1^2}\left[\dot{A}^2(0) + A^2(0)(\omega^2 + \dot{\alpha}^2(0)) + \dot{B}^2(0) + B^2(0)(\omega^2 + \dot{\beta}^2(0))\right] = const.$$

Two more integrals

$$u_1^2 - v_1^2 = C_1, \quad u_2^2 - v_2^2 = C_2, \tag{28}$$

where

$$C_1 = \frac{\dot{\alpha}^2(0)A^4(0)}{\omega^2}, \quad C_2 = \frac{\dot{\beta}^2(0)B^4(0)}{\omega^2} \tag{29}$$

are obtained by considering, correspondingly, the first-and-third and the second-and-fourth equations.

And, finally, from the first, fifth, and sixth equations, a more complicated solution is derived as

$$Gu_1 - \frac{Nu_1^2}{2} - \frac{\delta_1}{2\omega} L_1 L_2 \cos 2\theta = C_3 = const. \tag{30}$$

Here, the following notations are used:

$$G = \frac{1}{2\omega}\left[C_0(3\gamma_2 - 2\delta_1) + 2\Delta_1\right],$$
$$N = \frac{1}{2\omega}\left[3(\gamma_1 + \gamma_2) - 4\delta_2\right],$$
$$L_1 = \sqrt{(C_0 - u_1)^2 - C_2},$$
$$L_2 = \sqrt{u_1^2 - C_1}. \tag{31}$$

In general, the solutions determined allow a method of distribution of the initially added energy among the various modes of the shell oscillations to be traced. To determine a period of the energy transmitting from one mode to another, it is necessary to investigate the system of equations

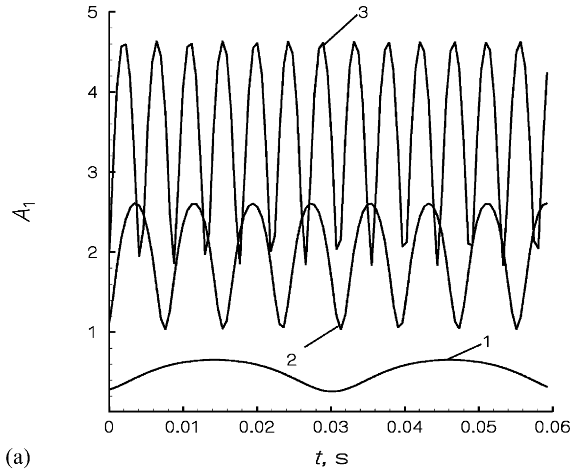
$$\frac{du_1}{dt} = -\frac{\delta_1}{2\omega} L_1 L_2 \sin 2\theta,$$
$$\frac{d\theta}{dt} = G - N u_1 - \frac{\delta_1}{2\omega L_1 L_2}[u_1 L_1^2 - L_2^2(C_0 - u_1)]\cos 2\theta, \qquad (32)$$

derived from Eq. (26) taking into account integrals (27) and (28).

Thus, the input problem concerned with interaction of the non-conjugated wave shapes was reduced to analysis of two equations (32), which were set up concerning the parameters of amplitude u_1 and phase θ. If the values of these parameters are determined, then the other parameters u_2, v_1, v_2, φ_1 and φ_2, involved in the averaged equations (26), will also be known. The phase velocities $\dot{\alpha}$ and $\dot{\beta}$ are defined by the formulae $\dot{\alpha} = \omega M_1/A^2$ and $\dot{\beta} = \omega M_2/B^2$.

To give a numerical illustration of the solution, consider a shell with parameters (20) and length-to-radius relation of $l = 2.442R$. In this case, two of its eigenfrequencies, which correspond to modes $m = 1$, $n_1 = 5$, and $m = 1$, $n_2 = 8$, are practically coincident with one another ($\omega_1 \approx \omega_2 \approx 337.5$ Hz). Figs. 3 and 4 show the results of integration of Eqs. (32) with initial conditions of two types: $\dot{\alpha}(0) = \dot{\beta}(0) = 0$ (Fig. 3) and $\dot{\alpha}(0) = \dot{\beta}(0) = \omega$ (Fig. 4). The initial conditions of the first and second types correspond to those cases, when deflection w from formula (23) represents a superposition of standing and running waves respectively. Curves 1 are obtained with $A(0) = 0.75h$, curves 2—with $A(0) = 1.5h$, and curves 3—with $A(0) = 2h$. In the figures, the notation $A_1 = u_1/h^2$ is used. Also, $\theta(0)=\pi/4$, $A(0) = B(0)$, $\dot{A}(0) = \dot{B}(0) = 0$ are accepted here. In accordance with relationships (25) and (28), the initial value of the function u_1 is of the form $u_1(0) = (1/2\omega_1^2)[\dot{a}^2(0) + a^2(0)(\omega_1^2 + \dot{\alpha}^2(0))]$. To clarify the results obtained, consider relationships (21), (25) and (28), (29). From these it follows that the energy exchange in the problem considered is, similar to the case of interaction of the conjugated forms, a time-periodic process or close to it. The "oscillations" of the amplitude parameter $A_1(t)$ characterizing the mode $m = 1$, $n_1 = 5$ are a pre-condition for the corresponding "oscillations" of $B_1(t) = u_2/h^2$ parameter of the second mode $m = 1$, $n_2 = 8$ and vice versa. Periodicity of amplitude parameters u_1 and u_2 causes a corresponding modification of parameters v_1 and v_2 which follows from formulae (28). Furthermore, periods of the "oscillations" depend essentially on the values of initial amplitudes $A(0)$, $B(0)$ and phase velocities $\dot{\alpha}(0), \dot{\beta}(0)$. These values determine the constants C_1 and C_2 (see formulae (29)) entering into Eqs. (28) and (32). If the $A(0)$ amplitude increases, periods of the $A(t)$ functions will decrease. This is evidence of the "hardening" type of geometrical non-linearity taken into account in the calculation model (22). It should be noted that such non-linearity is characteristic of this particular example. Another "softening" type of non-linearity, in which the shell's oscillation periods increase with increases in the amplitudes, is possible if one takes different parameters of the shell and other initial conditions.

It should be noted that the function $\dot{\theta}(t)$ has qualitatively different forms in the above two cases. For instance, if $\dot{\alpha}(0) = \dot{\beta}(0) = 0$, these functions are close to sinusoidal, whereas in the second case ($\dot{\alpha}(0) = \dot{\beta}(0) \neq 0$) their appearance is close to periodically repeating pulses.



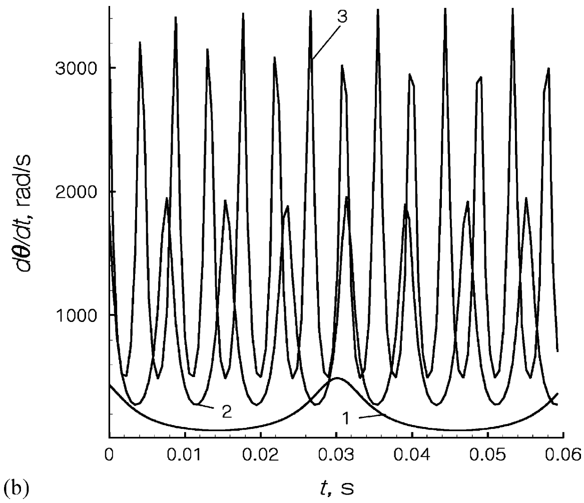


Fig. 3. Changes in time of the (a) amplitude A_1 and (b) phase velocity $d\theta/dt$ at $l/R = 2.442$ and 1—$A(0) = 0.75h$, 2—$A(0) = 1.5h$, 3—$A(0) = 2h$ in case $\dot{\alpha}(0) = \dot{\beta}(0) = 0$, $\theta(0) = \pi/4$, $A(0) = B(0)$.

Figs. 5 and 6 illustrate the impact of variation of the shell's parameters (in particular, its length l) on the amplitude $A(t)$ and phase velocity $\dot{\theta}(t)$. The results of numerical integration of Eqs. (32) are presented in Fig. 5. They were obtained with initial conditions $\dot{\alpha}(0) = \dot{\beta}(0) = 0$, and those presented in Fig. 6 with $\dot{\alpha}(0) = \dot{\beta}(0) = \omega$. Curves 1, 2, and 3 in both the figures are drawn for $l = 2.442R$, $l = 2.340R$, and $l = 2.450R$, respectively. The frequency "detune" $\bar{\Delta} = \omega_2 - \omega_1$ in each of these cases is equal to $\bar{\Delta}_1 = 0, \bar{\Delta}_2 = -3.1$ Hz and $\bar{\Delta}_3 = 1.4$ Hz.

Therefore, even a slight "detune" between the frequencies corresponding to different modes essentially affects the nature of the modes interaction, because the "detune" noticeably influences the values of $A_1(t)$ and $\dot{\theta}(t)$, which, in turn, affect the parameters $\varphi_1(t)$, $\varphi_2(t)$ and $B(t)$.

In Figs. 7 and 8, phase trajectories are shown, which have been drawn on the basis of Eqs. (32) under a trivial initial conditions (in this case $C_1 = 0$, $C_2 = 0$). Curves 1, 2, and 3 shown in Fig. 7 correspond to the similar curves presented in Fig. 3a, and those shown in Fig. 8 to the curves of Fig. 5a.

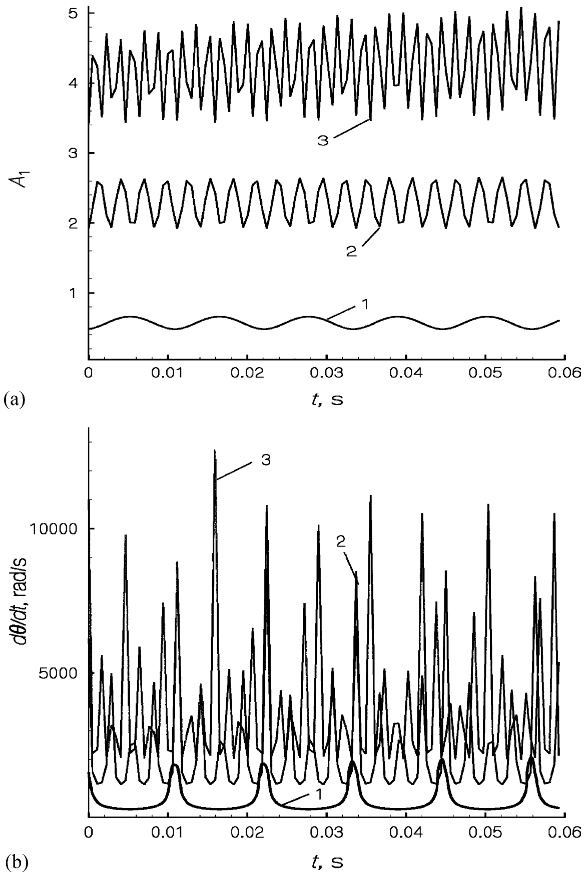


Fig. 4. Changes in time of the (a) amplitude A_1 and (b) phase velocity at $l/R = 2.442$ and 1—$A(0) = 0.75h$, 2—$A(0) = 1.5h$, 3—$A(0) = 2h$ in case $\dot{\alpha}(0) = \dot{\beta}(0) = \omega$, $\theta(0) = \pi/4$, $A(0) = B(0)$.

A form of the phase trajectories for initial conditions of the second case ($\dot{\alpha}(0) = \dot{\beta}(0) \neq 0$) is more complicated because of presence of pronounced beating in the $A_1(t)$ functions (see Figs. 4a and 6a).

4.2. The internal resonance $\omega_1 \approx 2\omega_2$

Investigation of the form of interaction in this case needs the non-linear terms of the fifth order to be accounted for in the equations system (22). Solution of this system will coincide in its form with Eq. (25), but here, instead of the $\psi_{1,2}$ phases, it is necessary to adopt

$$\psi_1 = 2(\omega t + \vartheta_1), \quad \psi_2 = 2\left(\frac{\omega t}{2} + \vartheta_2\right) \quad (\omega = \omega_1). \tag{33}$$

The first-approximation equations for determination of the u_i, v_i, ϑ_i ($i = 1{,}2$) parameters take the following form:

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = \frac{e_1 v_1 v_2^2}{4\omega} \cos 2\bar{\theta}, \quad \frac{\mathrm{d}u_2}{\mathrm{d}t} = -\frac{d_2 v_1 v_2^2}{2\omega} \cos 2\bar{\theta},$$

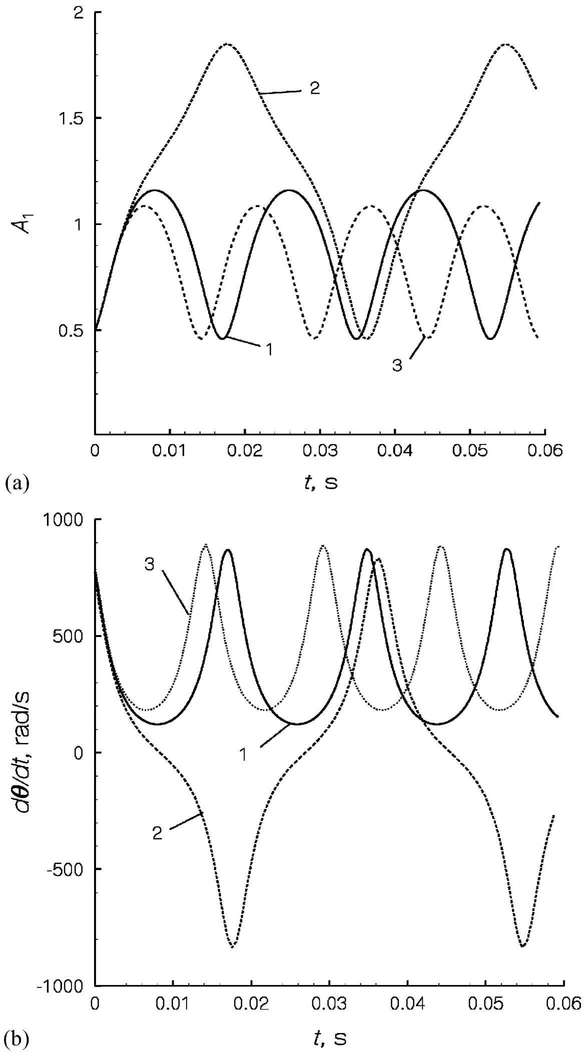
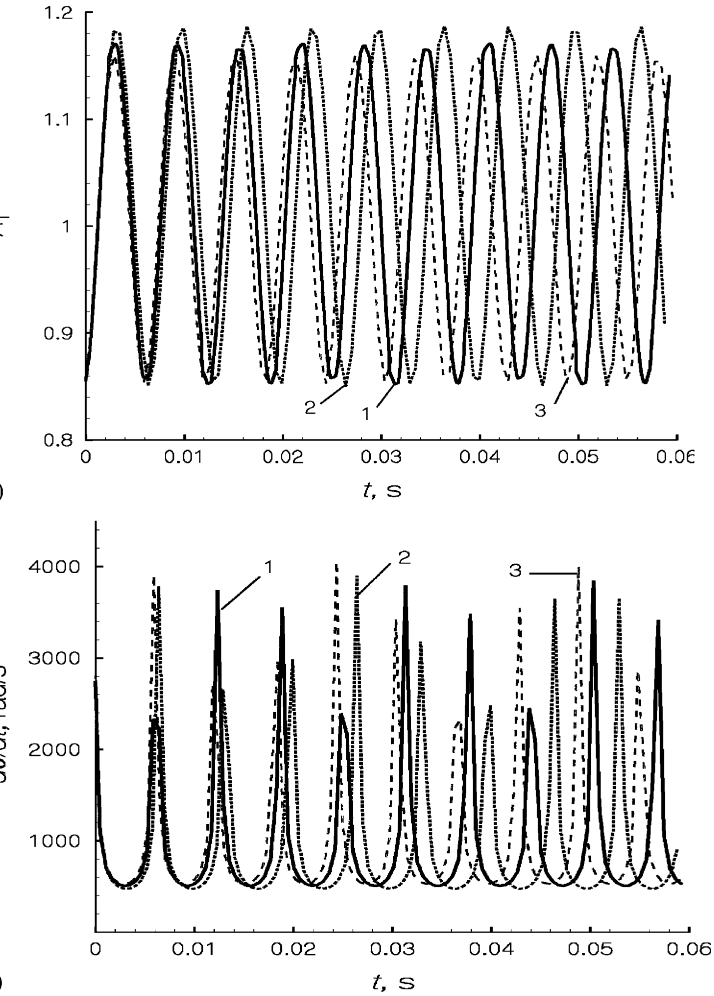


Fig. 5. Changes in time of the (a) amplitude A_1 and (b) phase velocity in case $A(0) = B(0) = h$, $\dot{\alpha}(0) = \dot{\beta}(0) = 0$, $\theta(0) = \pi/4$ at: 1—$l/R = 2.442$; 2—$l/R = 2.43$; 3—$l/R = 2.45$.

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = e_1 \frac{u_1 v_2^2}{4\omega} \cos 2\bar{\theta}, \quad \frac{\mathrm{d}v_2}{\mathrm{d}t} = -\frac{d_2 u_2 v_1 v_2}{2\omega} \cos 2\bar{\theta},$$

$$\frac{\mathrm{d}\vartheta_1}{\mathrm{d}t} = \frac{1}{2\omega}\left[\frac{3}{2}\gamma_1 u_1 + \delta_1 u_2 + c_1\left(2u_1^2 + \frac{v_1^2}{2}\right) + \frac{3}{2} d_1 u_1 u_2 + e_1\left(u_2^2 + \frac{v_2^2}{2} - \frac{u_1 v_2^2}{4v_1}\sin 2\bar{\theta}\right)\right],$$

$$\frac{\mathrm{d}\vartheta_2}{\mathrm{d}t} = \frac{1}{\omega}\left[\Delta_2 + \frac{3}{2}\gamma_2 u_2 + \delta_2 u_1 + c_2\left(2u_2^2 + \frac{v_2^2}{2}\right) + \frac{3}{2} d_2 u_1 u_2 + e_2\left(u_1^2 + \frac{v_1^2}{2} - \frac{u_2 v_1^2}{4v_2}\sin 2\bar{\theta}\right)\right], \quad (34)$$



(a)

(b)

Fig. 6. Changes in time of the (a) amplitude A_1 and (b) phase velocity in case $A(0) = B(0) = h$, $\dot{\alpha}(0) = \dot{\beta}(0) = \omega$, $\theta(0) = \pi/4$ at: 1—$l/R = 2.442$; 2—$l/R = 2.43$; 3—$l/R = 2.45$.

where

$$\bar{\theta} = \vartheta_1 - 2\vartheta_2, \quad \Delta_2 = \omega_2^2 - \omega_1^2/4. \tag{35}$$

In this case, from Eqs. (34), instead of Eq. (27) the following solution is obtained:

$$\frac{2u_1}{e_1} + \frac{u_2}{d_2} = C_0 = const. \tag{36}$$

Two more solutions will retain form (28). The fourth integral essentially differs from Eq. (30) and corresponds to the dependence

$$G_1 u_1 + \frac{G_2}{2} u_1^2 + \frac{G_3}{3} u_1^3 - \frac{e_1}{8\omega} L_2 L_1^2 \sin 2\bar{\theta} = C_3 = const, \tag{37}$$

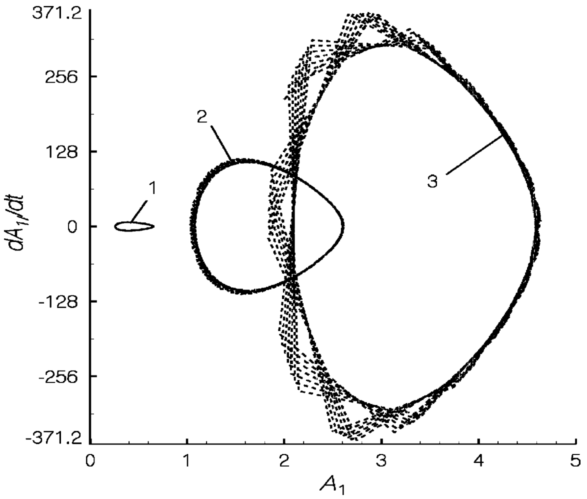


Fig. 7. Phase trajectories for the periodic vibrations of the shells at: 1—$A(0) = 0.75h$, 2—$A(0) = 1.5h$, 3—$A(0) = 2h$ in case $\dot{\alpha}(0) = \dot{\beta}(0) = 0$, $\theta(0) = \pi/4$, $l/R = 2.442$.

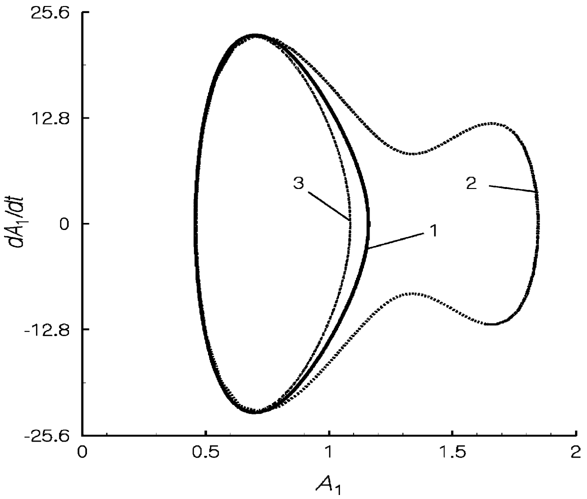


Fig. 8. Phase trajectories for the periodic vibrations of the shells at: 1—$l/R = 2.442$; 2—$l/R = 2.43$; 3—$l/R = 2.45$ in case $\dot{\alpha}(0) = \dot{\beta}(0) = 0$, $\theta(0) = \pi/4$, $A(0) = B(0) = h$.

with

$$
\begin{aligned}
G_1 &= \frac{1}{2\omega}\left[d_2^2 C_0^2\left(\frac{3}{2}\,e_2 - 10c_2\right) + (\delta_1 - 6\gamma_2)d_2 C_0 - \left(\frac{e_1}{2} - 2c_2\right)C_2 - \left(\frac{c_1}{2} - 2d_4\right)C_1 - 4\Delta_2\right],\\
G_2 &= \frac{1}{2\omega}\left[\frac{3}{2}\,\gamma_1 - 4\delta_2 - \frac{2d_2}{e_1}\,(\delta_1 - 6\gamma_2) - \frac{4C_0}{e_1}\,d_2^2\left(\frac{3}{2}\,e_1 - 10c_2\right) + 3C_0 d_2\left(\frac{d_1}{2} - 2d_2\right)\right],\\
G_3 &= \frac{1}{2\omega}\left[\frac{4d_2^2}{e_1^2}\left(\frac{3}{2}\,e_1 - 10c_2\right) + \frac{6d_2}{e_1}\left(2d_2 - \frac{d_1}{2}\right) + \left(\frac{5}{2}\,c_1 - 6e_2\right)\right].
\end{aligned} \tag{38}
$$

The time-dependant functions u_1 and $\bar{\theta}$ of Eq. (37) satisfy the following system of equations:

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = \frac{e_1 L_2 L_1^2}{4\omega}\cos 2\bar{\theta},$$
$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}t} = G_1 + G_2 u_1 + G_3 u_1^2 + G_4 \sin 2\bar{\theta}. \tag{39}$$

Here

$$G_4 = \frac{1}{8\omega L_2}\left(4d_2 L_2^2 u_2 - e_1 L_1^2 u_1\right), \quad u_2 = d_2\left(C_0 - \frac{2u_1}{e_1}\right),$$
$$L_1 = \sqrt{d_2^2\left(C_0 - \frac{2u_1}{e_1}\right)^2 - C_2}, \quad L_2 = \sqrt{u_1^2 - C_1}. \tag{40}$$

Some numerical results of analysis of Eqs. (39) are presented in Fig. 9. A shell with parameters (20) has been considered with its length being equal to $l = 2.415R$. Under this condition, a subharmonic resonance $\omega_1 \approx 2\omega_2$ taking place, with ω_1 frequency corresponding to the mode of $m = 1$, $n = 11$, and ω_2 frequency—to the mode of $m = 1$, $n = 7$.

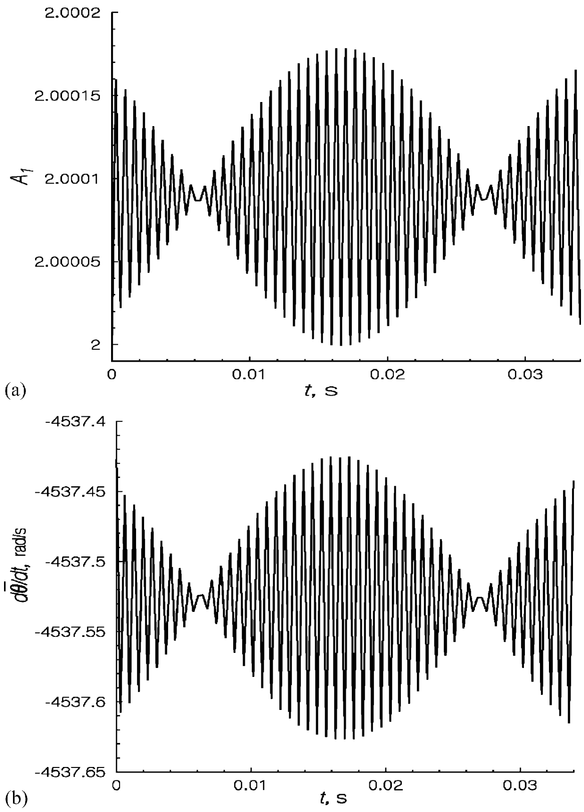


Fig. 9. Changes in time of the (a) amplitude A_1 and (b) phase velocity $\mathrm{d}\bar{\theta}/\mathrm{d}t$ at $A(0) = 2h$, $l/R = 2.415$, $\dot{\alpha}(0) = \dot{\beta}(0) = 0, \bar{\theta}(0) = \pi/4$.

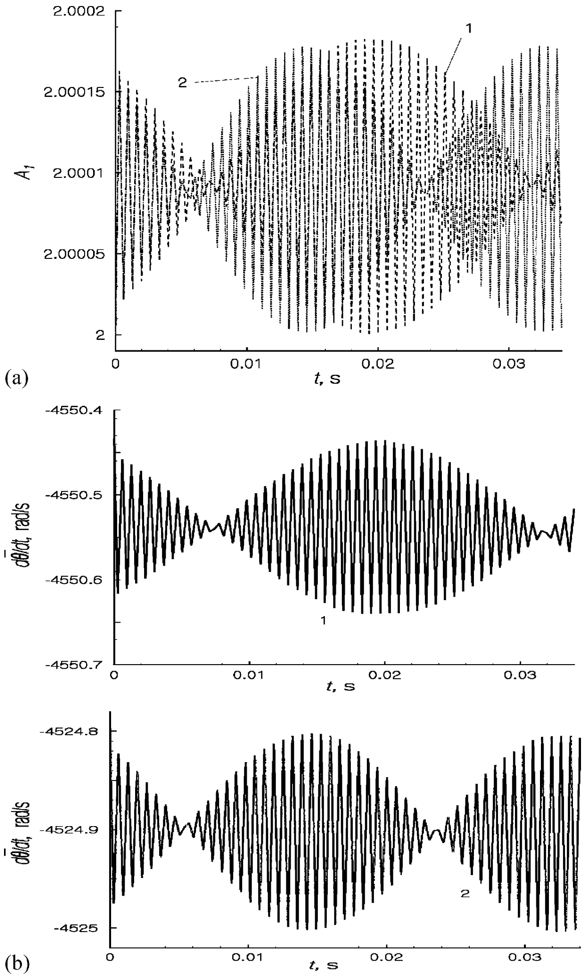


Fig. 10. Changes in time of the (a) amplitude A_1 and (b) phase velocity $d\bar{\theta}/dt$ at: 1—$l/R = 2.40$; 2—$l/R = 2.43$, when $A(0) = B(0) = 2h$, $\dot{\alpha}(0) = \dot{\beta}(0) = 0, \bar{\theta}(0) = \pi/4$.

Fig. 9 illustrates characteristic features of variation in time of the amplitude parameter A_1 and phase velocity $\dot{\bar{\theta}}$ for the case of $\dot{\alpha}(0) = \dot{\beta}(0) = 0$ for a free oscillating shell. As can be seen, the "oscillations" of $A_1(t)$ and $\dot{\bar{\theta}}(t)$ functions correspond to typical beating. The energy transmitting from one mode (m,n_1) to another (m,n_2) and vice versa will be, in some moments, realized completely (i.e., one of the forms will not be excited in these moments, whereas the other form will be characterized by a maximal deflection from the middle surface).

Fig. 10 shows the impact of the dimensionless parameter l/R to free vibrations of the shell; moreover curves 1 are for $l/R = 2.40$,and curves 2 for $l/R = 2.43$. It is seen clearly that the periods of beatings of $A_1(t)$ and $\dot{\bar{\theta}}(t)$ are essentially sensitive to some, even a slight shift of a shell from its intrinsic resonance.

Figs. 11 and 12 show phase patterns corresponding to the free periodical oscillations of the shell for the resonance under consideration. Fig. 11 is obtained for initial conditions of the

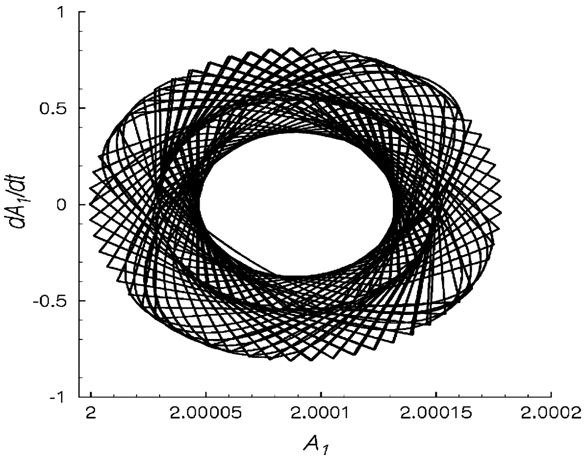


Fig. 11. Phase trajectories for the periodic vibrations of the shell having $l/R = 2.415$, $\dot{\alpha}(0) = \dot{\beta}(0) = 0, \bar{\theta}(0) = \pi/4$ when $A(0) = 2h$.

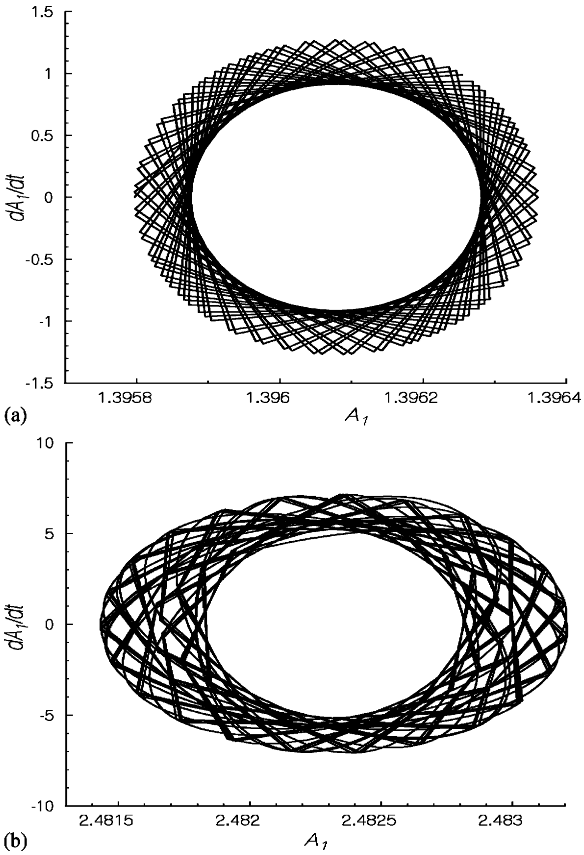


Fig. 12. Phase trajectories for the periodic vibrations of the shell having $l/R = 2.415$, $\dot{\alpha}(0) = \dot{\beta}(0) = \omega, \bar{\theta}(0) = \pi/4$ when (a) $A(0) = 1.5h$ and (b) $A(0) = 2\,h$.

type $\dot{\alpha}(0) = \dot{\beta}(0) = 0$, $A(0) = 2h$, $\bar{\theta}(0) = \pi/4$; Figs. 12(a) and (b) are for $A(0) = 1.5h$ and $A(0) = 2h$, respectively (in both cases it was accepted that $\dot{\alpha}(0) = \dot{\beta}(0) = \omega$, $\quad \bar{\theta}(0) = \pi/4$).

5. Conclusions

A method for the investigation of the free non-linear oscillations of cylindrical shells is suggested that accounts for interaction of the conjugated forms and forms of general type. The method is based upon the averaging-method concepts of Bogolyubov and Mitropolsky, which have been employed for the solution of equations of the special form (Eq. (22) type) describing the multi-mode oscillations of shells.

As a result of the investigations, it has been established that the effects of the interaction of the forms are manifest mostly in conditions of internal resonances. Construction of a number of first solutions became possible in the course of producing these resonances. These integrals have established definite 'energy-transmitting' connections between the various modes (within the regime of free oscillations of a shell). Employment of the solutions obtained enabled a solution of the initial system of high-degree dynamical equations to be reduced to the analysis of two equations constructed relative to the base variables: amplitude parameter u_1 of one of the modes and phase parameter θ (or $\bar{\theta}$).

Processes of the interaction of the modes essentially depend upon stated initial conditions for the functions $\dot{\alpha}(t)$ and $\dot{\beta}(t)$, occurring in expression (23) for deflection value w. In particular, these processes are qualitatively different in the cases of $\dot{\alpha}(0) = 0$ ($\dot{\beta}(0) = 0$) and $\dot{\alpha}(0) \neq 0$ ($\dot{\beta}(0) \neq 0$). This is seen if Figs. 3 with 4, 5 with 6, and 11 with 12 are compared. On the other hand, the character of initial conditions for $\alpha(t)$ and $\beta(t)$ determines whether the total deflection is a superposition of the traditional standing waves, or of the running waves. Variation of the initial values of amplitudes $A(0)$ and $B(0)$, in its turn, is a pre-condition for the corresponding variation of periods of the energy transmitting from one form to another.

In conclusion, it should be noted that the method suggested can be extended to the case of arbitrary number of the interacting modes. The only condition is that the eigenfrequencies of the modes have to meet the specific resonance relationships.

Appendix A

Time functions $\phi_j = \phi_i(t)$, $(j = 1-26)$ used in correlation (6) are given by

$$\phi_1 = \frac{E\lambda^2 f_1}{R\Delta(\lambda, s_1)}(1 - s_1^2 R f_5), \quad \phi_2 = \frac{E\lambda^2 f_2}{R\Delta(\lambda, s_1)}(1 - s_1^2 R f_5),$$

$$\phi_3 = \frac{E\lambda^2 f_3}{R\Delta(\lambda, s_2)}(1 - s_1^2 R f_5), \quad \phi_4 = \frac{E\lambda^2 f_4}{R\Delta(\lambda, s_2)}(1 - s_2^2 R f_5),$$

$$\phi_5 = \frac{E\lambda^2}{R\Delta(2\lambda,0)}[s_1^2 R(f_1^2 + f_2^2) + s_2^2 R(f_3^2 + f_4^2) - 4f_5],$$

$$\phi_6 = \frac{E\lambda^2 s_1^2 (f_2^2 - f_1^2)}{2\Delta(0,2s_1)}, \quad \phi_7 = \frac{E\lambda^2 s_2^2 (f_4^2 - f_3^2)}{2\Delta(0,2s_2)},$$

$$\phi_8 = -\frac{E\lambda^2 s_1^2 f_1 f_2}{\Delta(0,2s_1)}, \quad \phi_9 = -\frac{E\lambda^2 s_2^2 f_3 f_4}{\Delta(0,2s_2)},$$

$$\phi_{10} = \frac{E\lambda^2 s_1^2 f_1 f_5}{\Delta(3\lambda,s_1)}, \quad \phi_{11} = \frac{E\lambda^2 s_2^2 f_3 f_5}{\Delta(3\lambda,s_2)},$$

$$\phi_{12} = \frac{E\lambda^2 s_1^2 f_2 f_5}{\Delta(3\lambda,s_1)}, \quad \phi_{13} = \frac{E\lambda^2 s_2^2 f_4 f_5}{\Delta(3\lambda,s_2)},$$

$$\phi_{14} = \frac{E\lambda^2 (s_1 + s_2)^2}{4\Delta(2\lambda, s_1 - s_2)}(f_1 f_3 + f_2 f_4),$$

$$\phi_{15} = -\frac{E\lambda^2 (s_1 + s_2)^2}{4\Delta(0, s_1 - s_2)}(f_1 f_3 + f_2 f_4),$$

$$\phi_{16} = \frac{E\lambda^2 (s_1 - s_2)^2}{4\Delta(2\lambda, s_1 + s_2)}(f_1 f_3 - f_2 f_4),$$

$$\phi_{17} = -\frac{E\lambda^2 (s_1 + s_2)^2}{4\Delta(0, s_1 + s_2)}(f_1 f_3 - f_2 f_4),$$

$$\phi_{18} = \frac{E\lambda^2 (s_1 + s_2)^2}{4\Delta(2\lambda, s_1 - s_2)}(f_2 f_3 - f_1 f_4),$$

$$\phi_{19} = -\frac{E\lambda^2 (s_1 - s_2)^2}{4\Delta(0, s_1 - s_2)}(f_2 f_3 - f_1 f_4),$$

$$\phi_{20} = \frac{E\lambda^2 (s_1 - s_2)^2}{4\Delta(2\lambda, s_1 + s_2)}(f_1 f_4 + f_2 f_3),$$

$$\phi_{21} = \frac{E\lambda^2 (s_1 + s_2)^2}{4\Delta(0, s_1 + s_2)}(f_1 f_4 + f_2 f_3),$$

$$\phi_{22} = \frac{Ef_5}{128\lambda^2 R}, \quad \phi_{23} = -\frac{E\lambda^2 s_1^2 f_1 f_5}{\Delta(5\lambda,s_1)},$$

$$\phi_{24} = -\frac{E\lambda^2 s_1^2 f_2 f_5}{\Delta(5\lambda,s_1)}, \quad \phi_{25} = -\frac{E\lambda^2 s_2^2 f_3 f_5}{\Delta(5\lambda,s_2)},$$

$$\phi_{23} = -\frac{E\lambda^2 s_2^2 f_4 f_5}{\Delta(5\lambda,s_2)}.$$

Appendix B

In Eqs. (8) the frequencies ω_i^2 and constants K_{jk} are given by

$$\omega_1^2 = \frac{1}{\rho}\left[\frac{D}{h}\,\Delta(\lambda, s_1) + \frac{E\lambda^4}{R^2\Delta(\lambda, s_1)}\right],$$

$$\omega_2^2 = \frac{1}{\rho}\left[\frac{D}{h}\,\Delta(\lambda, s_2) + \frac{E\lambda^4}{R^2\Delta(\lambda, s_2)}\right],$$

$$\omega_3^2 = \frac{64}{35\rho}\left(\frac{8D\lambda^4}{h} + \frac{35}{64}\frac{E}{R^2}\right),$$

$$k_{11} = \frac{E}{16\rho}\,(\lambda^4 + 3s_1^4), \quad k_{12} = \frac{E}{8\rho}\,(s_1^2 s_2^2 + M),$$

$$k_{13} = -\frac{Es_1^2}{\rho R}\left[\frac{5}{8} + \frac{2\lambda^4}{\Delta(\lambda, s_1)}\right],$$

$$k_{14} = -\frac{E\lambda^4 s_1^4}{\rho}\left[\frac{1}{\Delta(\lambda, s_1)} + \frac{4}{\Delta(3\lambda, s_1)} + \frac{1}{\Delta(5\lambda, s_1)}\right],$$

$$k_{21} = k_{12}, \quad k_{22} = \frac{E}{16\rho}\,(\lambda^4 + 3s_2^4),$$

$$k_{23} = -\frac{Es_2^2}{\rho R}\left[\frac{5}{8} + \frac{2\lambda^4}{\Delta(\lambda, s_2)}\right],$$

$$k_{24} = \frac{E\lambda^4 s_2^4}{\rho}\left[\frac{1}{\Delta(\lambda, s_2)} + \frac{4}{\Delta(3\lambda, s_2)} + \frac{1}{\Delta(5\lambda, s_2)}\right],$$

$$k_{31} = \tfrac{16}{35}\,k_{13}, \quad k_{32} = \tfrac{16}{35}\,k_{23}, \quad k_{33} = \tfrac{32}{35}\,k_{14}, \quad k_{34} = \tfrac{32}{35}\,k_{24},$$

$$M = \frac{\lambda^4}{2}\left\{(s_1 - s_2)^4\left[\frac{1}{\Delta(2\lambda, s_1 + s_2)} + \frac{1}{\Delta(0, s_1 - s_2)}\right]\right.$$
$$\left. + (s_1 + s_2)^4\left[\frac{1}{\Delta(0, s_1 + s_2)} + \frac{1}{\Delta(2\lambda, s_1 - s_2)}\right]\right\}.$$

Appendix C

The coefficients $\gamma_i, \delta_i, c_i, d_i, \ell_i$ $(i = 1, 2)$, used in Eq. (22) are given by

$$\gamma_2 = k_{22} - \frac{k_{23}k_{32}}{\omega_3^2},$$

$$\delta_1 = k_{12} - \frac{k_{13}k_{32}}{\omega_3^2}, \quad \delta_2 = k_{21} - \frac{k_{23}k_{31}}{\omega_3^2},$$

$$c_2 = \frac{k_{32}}{\omega_3^4}(k_{23}k_{34} + k_{24}k_{32}),$$

$$d_1 = \frac{1}{\omega_3^4}[k_{13}(k_{32}k_{33} + k_{31}k_{34}) + 2k_{14}k_{31}k_{32}],$$

$$d_2 = \frac{1}{\omega_3^4}[k_{23}(k_{31}k_{34} + k_{32}k_{33}) + 2k_{24}k_{31}k_{32}],$$

$$l_1 = \frac{k_{32}}{\omega_3^4}(k_{13}k_{34} + k_{14}k_{32}), \quad l_2 = \frac{k_{31}}{\omega_3^4}(k_{23}k_{33} + k_{24}k_{31}).$$

Appendix D. Definitions of symbols

w	adiarl sagging of a shell
ϕ	stress function of the middle surface
ρ	density of shell material
R	middle surface radius
l	length of a shell
h	shell thickness
t	time
x, y	radial and circumferential co-ordinates
E	Young's modulus of shell
μ	Poisson coefficient
$\Delta(M,N)$	operator of the form $\Delta(M, N) = (M^2 + N^2)^2$
$\lambda = \lambda_m = m\pi/l$, $s_k = n_k/R$, $(k = 1, 2)$	wave-formation parameters
m	number of longitudinal waves
n_k	number of complete circumferential waves
ω_i	shell eigenfrequencies
D	the flexural rigidity of the shell $\left(= \frac{Eh^3}{12(1-\mu^2)}\right)$
∇^4	operator of the form $\nabla^4 = \nabla^2\nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$
$f_i(t)$	the generalized co-ordinates of the shells
a, b	amplitude of bi-mode vibration of the shell

ϑ_1, ϑ_2	phase parameters of the vibrations
A, B	amplitude of bending waves in shells
α, β	phase parameters of bending waves

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